ON CERTAIN MÖBIUS TYPE SOLUTIONS FOR THE n–BODY PROBLEM IN A POSITIVE SPACE FORM

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Abstract. We study here the Möbius type solutions for the n-body problem in a two dimensional positive space form $\mathbb{M}^2_R$. With methods of Möbius geometry and using the Iwasawa decomposition of the Möbius group of automorphisms $\mathbf{Mob}_2\left(\mathbb{M}^2_R\right)$, we state algebraic functional conditions for the existence of such type of solutions in $\mathbb{M}^2_R$. We mention some examples of these type of solutions.

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1. Introduction

We consider in this paper the problem of studying the motion of n point interacting particles of masses $m_1, \cdots, m_n$ moving on a positive two dimensional space form $\mathbb{M}^2_R$ under the action of a suitable (cotangent) potential.

This work is a little branch of the forgotten, unpublished, incomplete, but full of ideas work [18] with E. Pérez-Chavela, to whom the second author thanks the fact of introducing him in the study of the curved celestial mechanics without leaving his profile of geometer. We omit along all the document the term *curved* because, as in [19], the studied space has a non-euclidian metric. Also, we omit the term *intrinsic*, because when we have

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chosen the geometric structure on $\mathbb{M}_R^2$ as Riemann surface $\widehat{\mathbb{C}}$ with the canonical complex variables $(z, \bar{z})$ endowed with the conformal metric

$$ds^2 = \frac{4R^4\, dz d\bar{z}}{(R^2 + |z|^2)^2}, \tag{1}$$

we have given the corresponding differentiable structure in such coordinates to this space (see [9] for more details).

Following the methods of the geometric Erlangen program as in [14, 18, 19] for the space $\mathbb{M}_R^2$, we define the conic motions of the n–body problem in terms of the action of one dimensional subgroups of Möbius transformations group $\mathbf{Mob}_2\,(\mathbb{M}_R^2)$ associated to a suitable vector fields. By the method of matching the vector field in the Lie algebra associated to the corresponding subgroup, with the cotangent gravitational field, as we did in [5] [17], [18] and [19], we state in each case functional algebraic conditions (depending on the time t) which the solutions must hold in order to be one of such Möbius solutions.

We organize the paper as follows.

In section we state the Iwasawa decomposition of $\mathbf{Mob}_2\,(\mathbb{M}_R^2)$ that allows us to obtain the algebraic and geometric classification of all Möbius transformations in *elliptics, hyperbolics*, and *parabolics*. The action of such one parametric transformations will generate the corresponding conic curves of the Möbius geometry of $\mathbb{M}_R^2$.

In section 3 we state the equations of motion of the problem in complex coordinates as in [5], [17], [18], but they are obtained as in [19] .

In section 4 we define the Möbius elliptic solutions, thorough the action of suitable one-dimensional parametric subgroups of isometries in $SU(2)$ associated to suitable Killing vector fields in the Lie algebra $su(2)$. There, by matching the Killing vector fields with the gravitational one, we find the main algebraic conditions in order of having a Möbius elliptic solution. We give new examples of such type of solutions for this problem for which the geodesic circle and the southern and northern tropics play an important role.

In section 5 we define the so called Möbius hyperbolic solutions, via the action of the one-dimensional parametric subgroup of Möbius hyperbolic transformations showing that they correspond to some particular type of the homothetic orbits found in [3, 4]. We also give the functional algebraic conditions (depending on the time t) on the positions of the particles for having one of such solutions.

In section 6 we define the Möbius nilpotent parabolic solutions. As in the previous cases we give the necessary and sufficient functional algebraic conditions which such orbits must hold. We show that, as in [5, 19], such type of motions do no exist.

2. The group $\mathbf{Mob}_2\,(\widehat{\mathbb{C}})$ and the invariants of the Möbius geometry

We give an algebraic classification of the Möbius transformations defined on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{M}^2_R \cup \{\infty\}$, which corresponds to the Riemann sphere of radius R endowed with the metric (1). The reader interested in the details on all the objects shown in this section can see them in the aforementioned reference [18] or in [8, 13, 14, 15].

Definition 1. *A Möbius transformation is a fractional linear transformation $f_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$,*

$$f_A(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$, and the set of these automorphisms is the group denoted by $\mathbf{Mob}_2\,(\widehat{\mathbb{C}})$ and named the Möbius group.

Any Möbius transformation f_A is associated to some matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) = \{A \in \mathrm{GL}(2,\mathbb{C}) \,|\, \det A = 1\},$$

which defines an isomorphism between the groups $\mathbf{Mob}_2\,(\widehat{\mathbb{C}})$ and $\mathrm{SL}(2,\mathbb{C})\,/\{\pm I\}$.

The special unitary subgroup is

$$\mathrm{SU}(2) = \{A \in \mathrm{SL}(2,\mathbb{C}) \,|\; \bar{A}^T A = I\},$$

and each matrix $A \in \mathrm{SU}(2)$ has the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ (see [8]).

We also know from the Lie group theory that $\mathrm{SU}(2)$ is the maximal compact subgroup of $SL(2,\mathbb{C})$, and that the *group of proper isometries* of $\mathbb{M}^2_R$ is the quotient $\mathrm{SU}(2)\,/\,\{\pm I\}$.

2.1. The elliptic Möbius group.

The *Lie algebra* of $\mathrm{SU}(2)$ is the 3-dimensional real linear space

$$\mathrm{su}(2) = \{X \in \mathrm{M}(2,\mathbb{C}) \,|\; \bar{X}^T = -X,\ \mathrm{trace}\,(X) = 0\}$$

spanned by the basis of complex Pauli's spinor matrices,

$$\left\{ X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\},$$

which are Killing vector fields on the Lie group. Applying the exponential application to the lines tX_1, tX_2 and tX_3 into $\mathrm{SU}(2)$ we obtain the respective isometric one-dimensional subgroups,

1. The subgroup

$$\exp(t\,X_1) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

which defines the one-parametric family of acting Möbius transformations in $\widehat{\mathbb{C}}$,

$$f_1(t,z) = \frac{z\cos t + \sin t}{-z\sin t + \cos t}. \tag{2}$$

This flow has asociated the vector field $1+z^2$ in $\widehat{\mathbb{C}}$ obtained when we derive (2) respect to the parameter t and evaluate it at $t=0$. This vector field corresponds to the complex differential equation

$$\dot{z} = 1 + z^2, \tag{3}$$

whose solutions correspond to coaxal circles with fixed points $z_1 = i$ and $z_2 = -i$.

Such that flow defines a foliation of $\mathbb{M}_R^2$, which divides it in two connected components (hemispheres) with a common border in the geodesic separatrix $\operatorname{Im} z = 0$ (meridian circle). Each component is foliated by the circular periodic orbits, solutions of the differential equation (3) and the fixed points are the centers of such foliations, also called *focus* of the whole set of circles in the sense of the Möbius geometry of $\mathbb{M}_R^2$, as in [14]. They can be seen in the sphere $\mathbb{S}_R^2$, up a suitable rotation, as two sets of periodic concentric solutions, one sited inside of the north hemisphere bounded by the geodesic separatrix (equator), and other one sited inside of the south hemisphere bounded also by such geodesic (see [18] for details).

2. The subgroup

$$\exp(t\,X_2) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},$$

which defines the one-parametric family of acting Möbius transformations

$$f_2(t,z) = e^{2it}\,z. \tag{4}$$

This flow is associated to the vector field iz in $\widehat{\mathbb{C}}$ and corresponds to the complex differential equation,

$$\dot{z} = 2i\,z. \tag{5}$$

The orbits of the action of the one-parametric subgroup $\{\exp(tX_2)\}$ in $\mathbb{M}_R^2$ are circular orbits with center in fixed point $z=0$.

3. The subgroup

$$\exp(t\,X_3) = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix},$$

which defines the one-parametric family of acting Möbius transformations

$$f_3(t,z) = \frac{z\cos t + i\sin t}{z\,i\sin t + \cos t}. \tag{6}$$

This flow is associated to the vector field $i(1-z^2)$ in $\widehat{\mathbb{C}}$ which corresponds to the complex differential equation

$$\dot{z} = i(1-z^2), \tag{7}$$

whose orbits correspond also to coaxial circles with focus in the fixed points $z=-1$ and $z=1$.

Also, as in the first case, this flow defines a foliation of $\mathbb{M}^2_R$, which divides it in two connected components (hemispheres) with a common border in the geodesic separatrix $\mathrm{Re}\, z = 0$ (see [18] for details).

We will use the above one-dimensional subgroups for obtaining the whole set of the so called *Möbius elliptic solutions* (or, as in the dynamical literature: *Relative equilibria solutions*) of the n-body problem in $\mathbb{M}^2_R$.

2.2. **The hyperbolic normal Möbius group.** In the Lie algebra $sl(2,\mathbb{C})$ we have the hyperbolic vector field,

$$X_4 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we consider also the straight line $\{tX_4\}$ in $sl(2,\mathbb{C})$, it is applied under the exponential map into the one-parametric subgroup of transformations associated to the normal hyperbolic one dimensional subgroup of Möbius matrices of $\mathbf{Mob}_2\,(\widehat{\mathbb{C}})$ of the form

$$G_h(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

The above matrices generate the acting one dimensional parametric subgroup of Möbius transformations given by

$$f_{G_h(t)}(z) = e^t\, z. \tag{8}$$

The flow of this group is associated with vector field z in $\widehat{\mathbb{C}}$, and with the first order complex differential equation

$$\dot{z} = z \tag{9}$$

Such that flow is a set of straight lines arising in the origin of coordinates with fixed point $z=0$.

We will use this one-dimensional subgroup for obtaining the set of the *Möbius normal hyperbolic solutions* (or, in the dynamical literature: *homothetic solutions*) of this problem.

2.3. **The nilpotent parabolic Möbius group.** If in the Lie algebra $sl(2,\mathbb{C})$ we consider the nilpotent parabolic vector field,

$$X_5 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

the straight line $\{tX_5\}$ in $sl(2,\mathbb{C})$ is applied under the exponential map into the one-parametric subgroup of transformations associated to the nilpotent parabolic subgroup of Möbius matrices of $\mathbf{Mob}_2(\widehat{\mathbb{C}})$ of the form

$$G_p(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

which defines the subgroup of acting Möbius transformations

$$f_{G_p(t)}(z) = z + t, \tag{10}$$

in $\mathbb{M}_R^2$.

The one parametric subgroup (10) is associated to the unitary vector field 1 in $\widehat{\mathbb{C}}$, which defines the first order complex differential equation in $\mathbb{M}_R^2$,

$$\dot{z} = 1. \tag{11}$$

The flow is a set of horizontal parallel straight lines and there are not fixed points in $\mathbb{M}_R^2$.

As before, we will use this one-dimensional subgroup for obtaining the set of *Möbius nilpotent parabolic solutions* (or, in the dynamical literature: *parabolic solutions*).

2.4. **The Iwasawa decomposition of $\mathbf{SL}(2,\mathbb{C})$.** In the Möbius geometry of $\mathbb{M}_R^2$ (see [14, 15] for details), we have the following distinguished subgroups:

- $K = SU(2)$ called the *cyclic elliptic subgroup* of $\mathbf{SL}(2,\mathbb{C})$,
- $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$ named the *nilpotent parabolic subgroup* of $\mathbf{SL}(2,\mathbb{C})$,
- $B = \left\{ \begin{pmatrix} e^{\frac{\alpha}{2}} & 0 \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$ called the *normal hyperbolic subgroup* of $\mathbf{SL}(2,\mathbb{C})$.

The Iwasawa decomposition Theorem states that the Lie group $\mathbf{SL}(2,\mathbb{C})$ can be factorized by the before subgroups,

$$\mathbf{SL}(2,\mathbb{C}) = B\,N\,K. \tag{12}$$

This is, for any $A \in \mathbf{SL}(2,\mathbb{C})$, there exist unique matrices $P \in B, R \in N, S \in K$ such that A can be factorized in the form $A = PRS$ (see [12], [13] and [14] for more details).

An immediate result of this Theorem is also the following one.

Corollary 1. *For any Möbius transformation $f_A \in \mathbf{Mob}_2(\widehat{\mathbb{C}})$ there exist unique matrices $P \in B, R \in N, S \in K$ as in Iwasawa's Theorem, such that f_A can be factorized in the form $f_A = f_P \circ f_R \circ f_S$.*

Another important result for the work is the following.

Corollary 2. *Any one-dimensional parametric subgroup $g(t) \in \mathbf{Mob}_2(\widehat{\mathbb{C}})$ can be written in a unique way of the form*

$$g(t) = f_{P(t)} \circ f_{R(t)} \circ f_{S(t)} \tag{13}$$

with,

$$P(t) = \begin{pmatrix} a(t) & b(t) \\ -\bar{b}(t) & a(t) \end{pmatrix}, \quad R(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \quad S(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

3. Equations of motion and the Möbius solutions

In [17] the authors obtain the equations of motion for this problem through the stereographic projection of the sphere (of radius R) embedded in $\mathbb{R}^3$ into the complex plane $\mathbb{C}$ endowed with the metric (1). After, the authors in [19] using the Vlasov-Poisson equation, obtain the classical equation for the motion of particles with positives masses $m_1, m_2, \cdots, m_n$ in a Riemannian or semi-Remannian manifold with coordinates $(x^1, x^2, \cdots, x^N)$ endowed with a metric (g_{ij}) and associated connection Γ^i_{jk}, which are moving under the influence of a pairwise acting potential U. Such equations are given by

$$\frac{D\dot{x}^i}{dt} = \ddot{x}^i + \sum_{l,j} \Gamma^i_{lj} \dot{x}^l \dot{x}^j = \sum_k m_k g^{ik} \frac{\partial U}{\partial x^k}, \tag{14}$$

for $i = 1, 2, \cdots, N$, where $\dfrac{D}{dt}$ denotes the covariant derivative and $g^{-1} = (g^{ik})$ is the inverse matrix for the metric g.

Remark 1. We observe that in equation (14), the left hand side corresponds to the equations of the geodesic curves, whereas the right hand side corresponds to the gradient of the potential in the given metric. This means also that if the potential is constant, then the particles move along geodesics.

It is well known from the Beltrami-Bers' Theorem (see [7, 8, 9]) that all connected smooth two dimensional manifold admits an atlas such that its associated Riemannian metric is conformal and endows it with a Riemann surface structure. The Poincaré-Koebe Uniformization Theorem implies that all orientable connected smooth two dimensional manifold admits a Riemannian metric with negative constant Gaussian curvature (hyperbolic structure), except the Sphere $\mathbb{S}^2$ and the torus $\mathbb{T}^2$ (see [11, 9]). The obstruction for the existence of hyperbolic structure of the last mentioned surfaces is, from the Gauss-Bonnet Theorem, their Euler's characteristics. In other words, almost all the surfaces are hyperbolic [11].

Definition 2. *A two dimensional positive space form is a smooth connected surface with positive constant Gaussian curvature.*

The Minding's Theorem states that all the two dimensional manifolds with the same constant Gaussian curvature are locally isometric (see [7]). Therefore, any two dimensional space form is locally characterized up an

isometry by the sign of the curvature and it is locally isometric to one of the standard space forms: The plane with zero Gaussian curvature, the sphere of radius R with positive Gaussian curvature $K = \dfrac{1}{R^2}$ and the hyperboloid of radius R with negative Gaussian curvature $K = -\dfrac{1}{R^2}$. In particular, any positive space form belongs to the isometric class of $\mathbb{M}^2_R$ (and therefore to the same differentiable class [8], chapter 3).

We state in this section the equations of motion for the n–body problem in $\mathbb{M}^2_R$ chosen as the representation of the isometric class of the whole set of two dimensional positive space forms.

Let us denote by $\mathbf{z} = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$ the total vector position of n particles with masses $m_i > 0$ located in the point z_i on the space $\mathbb{M}^2_R \equiv \mathbb{C}$.

For the pair of points z_k and z_j in $\mathbb{M}^2_R$ we denote the geodesic distance between them by $d(z_k, z_j) = d_{kj}$ and define (see [17]) the cotangent relation

$$\cot_R\left(\frac{d_{kj}}{R}\right) = \frac{2(z_k\bar{z}_j + z_j\bar{z}_k)R^2 + (|z_k|^2 - R^2)(|z_j|^2 - R^2)}{4R^2\,|z_j - z_k|^2\,|R^2 + \bar{z}_j z_k|^2}. \tag{15}$$

The singular set in $\mathbb{M}^2_R$ for the n–body problem is the zero set of the equation

$$|z_j - z_k|^2\,|R^2 + \bar{z}_j z_k|^2 = 0.$$

From here, we obtain the following singular sets:

(1) The *singular collision set* given by $\Delta(C) = \cup_{kj}\,\Delta(C)_{kj}$, where the set

$$\Delta(C)_{kj} = \{\mathbf{z} = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid z_k = z_j,\ k \neq j\} \tag{16}$$

is the one obtained by the pairwise collision of the particles with masses m_j and m_k.

(2) The *singular geodesic conjugated set* given by $\Delta(A) = \cup_{kj}\,\Delta(A)_{kj}$, where the set

$$\Delta(A)_{kj} = \left\{\mathbf{z} = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid z_k = \frac{-R^2}{|z_j|^2}\,z_j,\ k \neq j\right\}, \tag{17}$$

is the one obtained by the pairwise antipodal points with masses m_j and m_k. We remark that the singularity holds in this case because such antipodal points are in reality geodesic conjugated points for an infinity of geodesic curves which indetermine the acting force of the potential.

It is clear that the presence of geodesic conjugated points in arbitrary equations of motion (14) on manifolds allows us to singularities of such mechanical system, by the same reason as in this particular case.

We define the *total singular set* of the problem as

$$\Delta = \Delta(C) \cup \Delta(A). \tag{18}$$

Let $\mathbf{z} = (z_1, z_2, \cdots, z_n)$ be the total vector position of n point particles with masses $m_1, m_2, \cdots, m_n > 0$ in the space $\mathbb{M}^2_R$, moving under the action of the potential

$$\begin{aligned} U_R &= U_R(\mathbf{z}, \bar{\mathbf{z}}) = \frac{1}{R} \sum_{1 \le k < j \le n}^{n} m_k m_j \cot_R \left(\frac{d_{kj}}{R} \right) \\ &= \frac{1}{R} \sum_{1 \le k < j \le n}^{n} m_k m_j \frac{2(z_k \bar{z}_j + z_j \bar{z}_k) R^2 + (|z_k|^2 - R^2)(|z_j|^2 - R^2)}{2R\,|z_j - z_k|\,|R^2 + \bar{z}_j z_k|}, \end{aligned} \tag{19}$$

defined in the set $(\mathbb{M}^2_R)^n \setminus \Delta$.

A direct substitution of the equations for the geodesics curves in $\mathbb{M}^2_R$ and the gradient

$$\frac{\partial U_R}{\partial \bar{z}_k} = \sum_{j=1, j \ne k}^{n} \frac{m_k m_j\,(R^2 + |z_k|^2)(|z_j|^2 + R^2)^2 (R^2 + \bar{z}_j z_k)(z_j - z_k)}{4R^2\,|z_j - z_k|^3\,|R^2 + \bar{z}_j z_k|^3} \tag{20}$$

for $k = 1, 2, \cdots, n$, in equation (14), shows that the solutions of the problem must satisfy the following system of second order ordinary differential equations

$$m_k \ddot{z}_k - \frac{2 m_k \bar{z}_k \dot{z}_k^2}{R^2 + |z_k|^2} = \frac{2}{\lambda(z_k, \bar{z}_k)} \frac{\partial U_R}{\partial \bar{z}_k}(\mathbf{z}, \bar{\mathbf{z}}), \tag{21}$$

where,

$$\lambda(z_k, \bar{z}_k) = \frac{4R^4}{(R^2 + |z_k|^2)^2} \tag{22}$$

is the value of conformal function for the Riemannian metric (1) in the point $(z, \bar{z})$ (see [17]).

Definition 3. *A Möbius solution for the n–body problem in $\mathbb{M}^2_R$ is a solution $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ of the equations of motion (21) which is invariant under some one-dimensional subgroup $\{G(t)\} \subset \mathbf{Mob}_2\,(\widehat{\mathbb{C}})$.*

Due to the Iwasawa decomposition (13) in Corollary 2, in the following sections we will study the Möbius solutions of (21) corresponding to the subgroups K, N and B of $\mathbf{Mob}_2\,(\mathbb{M}^2_R)$ which allow us to obtain a suitable classification of motions for the n–body problem in a positive space form.

4. Möbius elliptic solutions

We start our analysis of the Möbius solutions with the so called *elliptic solutions*, obtained by the action of one dimensional parametric subgroups of the third factor $SU(2)$ in the Iwasawa decomposition (13), some of which in fact have been studied in the papers [3, 4, 17, 18].

Definition 4. *A Möbius elliptic solution for n–body problem (21) is a solution $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ of the equations of motion (21) which is invariant to any one-dimensional subgroup $\{A(t)\} \subset SU(2)/\{\pm I\}$.*

Since the basic one-dimensional parametric subgroups (2), (4) and (6) generate under the composition of functions all one-dimensional parametric subgroups $\{A(t)\} \subset SU(2)/\{\pm I\}$, we have the following result obtained in [18].

Theorem 1. *Let be n point particles with masses $m_1, m_2, \cdots, m_n > 0$ moving in $\mathbb{M}^2_R$. An equivalent condition for $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ to be a Möbius elliptic solution of (21) is that the coordinates satisfy one of the following rational functional equations depending of the time.*

a. *The one obtained by the action of (2), that is,*

(23)
$$\frac{16R^6\,(1+z_k^2)(R^2 z_k - \bar{z}_k)}{(R^2+|z_k|^2)^4} = \sum_{j=1, j\neq k}^{n} \frac{m_j\,(|z_j|^2+R^2)^2(R^2+\bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3\,|R^2+\bar{z}_j z_k|^3}$$

with velocity $\dot{z}_k = 1 + z_k^2$ at each point.

b. *That obtained by the action of (4), that is,*

$$\frac{32\,R^6(|z_k|^2 - R^2)z_k}{(R^2+|z_k|^2)^4} = \sum_{j=1, j\neq k}^{n} \frac{m_j\,(|z_j|^2+R^2)^2(R^2+\bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3\,|R^2+\bar{z}_j z_k|^3} \tag{24}$$

with velocity $\dot{z}_k = 2i\,z_k$ at each point.

c. *The one obtained by the action of (6), that is,*

(25)
$$\frac{16R^6\,(1-z_k^2)(\bar{z}_k + R^2 z_k)}{(R^2+|z_k|^2)^4} = \sum_{j=1, j\neq k}^{n} \frac{m_j\,(|z_j|^2+R^2)^2(R^2+\bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3\,|R^2+\bar{z}_j z_k|^3}$$

with velocity $\dot{z}_k = i(1 - z_k^2)$ at each point.

Proof. For the first case, by straightforward computations, we have from equation (3) the equality

$$\ddot{z}_k = 2\,z_k\left(1+z_k^2\right), \tag{26}$$

which when is substituted, together with (3) into equation (21), gives us the relation (23). The proofs for the other cases are similar. □

The following result give us conditions on the initial positions of the particles for generating an elliptic solution of equation (24). This result is equivalent to that obtained in [1] for the so called fixed points.

Corollary 3. *With the hypotesis of Theorem 1, a necessary and sufficient condition for the initial positions $z_{1,0}, z_{2,0}, \cdots, z_{n,0}$ generating a Möbius elliptic solution for the system (21), invariant under the Killing vector field*

(4), is that the coordinates satisfy the following system of algebraic equations,

(27)
$$\frac{32\,R^6(|z_{k,0}|^2-R^2)z_{k,0}}{(R^2+|z_{k,0}|^2)^4}=\sum_{j=1,j\neq k}^{n}\frac{m_j\,(|z_{j,0}|^2+R^2)^2(R^2+\bar{z}_{j,0}z_{k,0})(z_{j,0}-z_{k,0})}{|z_{j,0}-z_{k,0}|^3\,|R^2+\bar{z}_{j,0}z_{k,0}|^3}$$

and the velocity in each particle is given by the relation $\dot{z}_{k,0}=2iz_{k,0}$*, for* $k=1,2,\cdots,n$*.*

Proof. Let $w_k=w_k(t)=e^{2it}\,z_{k,0}$ be the action of the Killing vector field (4) in the initial condition point $z_{k,0}$, with velocity $\dot{z}_{k,0}=2iz_{k,0}$. If we multiply equation (27) by the number e^{2it} in both sides, and use the equality $\bar{w}_j(t)w_k(t)=\bar{z}_{j,0}e^{-it}\,z_{k,0}e^{it}=\bar{z}_{j,0}\,z_{k,0}$, then we obtain the system,

(28)
$$\frac{32\,R^6(|w_k|^2-R^2)w_k}{(R^2+|w_k|^2)^4}=\sum_{j=1,j\neq k}^{n}\frac{m_j\,(|w_j|^2+R^2)^2(R^2+\bar{w}_jw_k)(w_j-w_k)}{|w_j-w_k|^3\,|R^2+\bar{w}_jw_k|^3},$$

which shows that $w_k(t)$ is a solution of (24).

The converse claim follows directly if in system (24) we put $t=0$. This proves the Corollary. □

In [3, 4] the authors show examples of Möbius elliptic solutions invariant under the Killing vector field (5) on the sphere embedded in $\mathbb{R}^3$, but when all the particles are sited for all time on the same circle obtained of intersecting with an horizontal plane. In [17] you can find examples in the two and three body problems defined on $\mathbb{M}^2_R$.

The following result, which is our version of the principal axis Theorem of Euler in $\mathbb{M}^2_R$, shows that such that for studyng the whole set of elliptic Möbius solutions it is sufficient with studying those invariant under the Killing vector field (5).

Lemma 1. *The solutions of equations of motion (21) invariants under the Killing vector fields (3) and (7) can be carried isometrically into those solutions of (21) invariant under the Killing vector field (5).*

Proof. For this is sufficient with showing that the Killing vector (3) and (7) are conjugated with the one (5).

We give the proof for the Killing vector fields (3) and (5). The other case follows in the same way.

Consider an element

$$A=\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}\in \mathrm{SU}(2) \tag{29}$$

and the corresponding isometric Möbius transformation in $\mathbb{M}^2_R$,

$$w=\frac{az+b}{-\bar{b}z+\bar{a}} \tag{30}$$

which carries the foliation of solutions of the Killing vector field (2) into those solutions of the isometric vector field (4).

This is, such that transformation carries the Killing vector field $\dot{z} = 2iz$ into the one $\dot{w} = 1 + w^2$, if and only if,

$$\frac{2iz}{(-\bar{b}z+\bar{a})^2} = \frac{\dot{z}}{(-\bar{b}z+\bar{a})^2} = \dot{w} = 1 + w^2 = \frac{(az+b)^2 + (-\bar{b}z+\bar{a})^2}{(-\bar{b}z+\bar{a})^2}, \tag{31}$$

which, for $-\bar{b}z + \bar{a} \neq 0$, allows us to the equation

$$2iz = (a^2 + \bar{b}^2)z^2 + 2(ab - \bar{a}\bar{b})z + (b^2 + \bar{a}^2). \tag{32}$$

Since equation (32) must hold for all z in and open subset of $\mathbb{C}$ then we obtain the pair of equations

$$\begin{array}{rcl} a^2 + \bar{b}^2 & = & 0 \\ ab - \bar{a}\bar{b} & = & i. \end{array}$$

(33)

The factors $a - i\bar{b} = 0$ and $a + i\bar{b} = 0$ in the first equation of the system (33) can be verified are equivalent to $w(0) = i$ and $w(\infty) = -i$. If we multiply by a in both sides of the second equation of compatibility in the same system (33) and use the first equation of that system and the equality $|a|^2 + |b|^2 = 1$, it becomes into the first factor $ia + \bar{b} = 0$.

If we put $a = \alpha_1 + i\alpha_2$ and $b = \beta_1 + i\beta_2$, then such that equations implies that $b = \alpha_2 + i\alpha_1$.

Therefore, the searched Möbius transformation has associated any rotation matrix

$$A = \left(\begin{array}{cc} \alpha_1 + \alpha_2 i & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_1 - \alpha_2 i \end{array} \right) \in \mathrm{SU}(2), \tag{34}$$

which can be seen as an element in $SO(2) \subset \mathrm{SU}(2)$ with $\alpha_1^2 + \alpha_2^2 = \dfrac{1}{2}$.

Conclusion is given by the Existence and Uniqueness Theorem for the complex differential equation associated to the given vector field (5) with initial conditions in the fixed points $w = -i$ and $w = i$.

This ends the proof. □

Remark 2. The above method in Lemma 1 will be employed for proving the existence of Möbius solutions invariant under the Killing vector fields (3) and (7), by carrying isometrically them into the Killing vector field (5).

4.1. Examples of Möbius elliptic solutions in the n-body problem.

We show here several types of solutions for the n-body problem by using the algebraic system (27). This is, for the initial real positions $z_k(0) = z_{k,0}$ and with velocities $\dot{z}_{k,0} = 2i\, z_{k,0}$ we construct, by using the Corollary 3, periodic circular orbits for obtaining solutions which generalize those obtained in [17] for equal masses.

From here now on in this section, the words *inside of the geodesic circle* $|z| = R$ means that in the sphere of radius R embedded in $\mathbb{R}^3$ the

corresponding motion is realized in the south-hemisphere while the words *outside of the geodesic circle* $|z| = R$ means that the corresponding motion is realized in the north-hemisphere (see [18]). On other hand, the word *degenerated solution* means that for small perturbation on the mass ratio the system changes on the number of Möbius elliptic solutions close to such that solution. Moreover, this property of *non-degeneracy* of the solutions must be related to the *stability* of the solutions in the sense of [2], since they are obtained under conditions of transversality (see [10]).

We will use the following notation along this section for short in the results. For $z \in \mathbb{M}_R^2$ we denote by

a. P_S to the *south pole* $z = 0$.
b. T_S to the *southern tropic* $|z| = (\sqrt{2} - 1)R$.
c. E_R to the *geodesic circle* $|z| = R$.
d. T_N to the *northern tropic* $|z| = \dfrac{R}{\sqrt{2} - 1}$
e. P_N to the *north pole* $z = \infty$.
f. Ω_1 to the open region from P_S to the southern tropic T_S,

$$\Omega_1 = \{z \in \mathbb{M}_R^2 \quad | \quad 0 < |z| < (\sqrt{2} - 1)R\}.$$

g. Ω_2 to the open region from T_S to the geodesic circle E_R,

$$\Omega_2 = \{z \in \mathbb{M}_R^2 \quad | \quad (\sqrt{2} - 1)R < |z| < R\}.$$

h. Ω_3 to the open region from E_R to the northern tropic T_N,

$$\Omega_3 = \{z \in \mathbb{M}_R^2 \quad | \quad R < |z| < \frac{R}{\sqrt{2} - 1}\}.$$

i. Ω_4 to the open region from T_N to the north pole P_N,

$$\Omega_4 = \{z \in \mathbb{M}_R^2 \quad | \quad \frac{R}{\sqrt{2} - 1} < |z|\}.$$

4.1.1. **New examples of Möbius elliptic solutions in the two-body problem in $\mathbb{M}_R^2$.** For the two-body case, we will show solutions of (21) invariant under the vector field (5) such that the two masses move along two different circles.

Firstly we observe that the system (27) for the two body problem becomes into the simple algebraic system (not depending on the time t),

$$\begin{aligned}
\frac{32\,R^6(|z_1|^2 - R^2)z_1}{(R^2 + |z_1|^2)^4} &= \frac{m_2\,(|z_2|^2 + R^2)^2(R^2 + \bar{z}_2 z_1)(z_2 - z_1)}{|z_2 - z_1|^3\,|R^2 + z_1\bar{z}_2|^3}, \\
\frac{32\,R^6(|z_2|^2 - R^2)z_2}{(R^2 + |z_2|^2)^4} &= \frac{m_1\,(|z_1|^2 + R^2)^2(R^2 + \bar{z}_1 z_2)(z_1 - z_2)}{|z_1 - z_2|^3\,|R^2 + \bar{z}_1 z_2|^3},
\end{aligned}$$
(35)

By Corollary 3, a necessary and sufficient condition for the existence of a Möbius solution invariant under the Killing vector field (4) is that the initial conditions $z_1(0) = \alpha$, $z_2(0) = \beta$ satisfy equations (35).

Moreover, it can be seen that in equations (35), $z_1 = \alpha$ is a real number, if and only if, $z_2 = \beta$ is also a real number (see [17, 18]). We obtain the following result.

Lemma 2. *Let $0 < \alpha < R$ be a real number. For the two body problem in $\mathbb{M}^2_R$ with equal masses, a necessary condition for the existence of one Möbius elliptic solution invariant under the Killing vector field (4), is that, for the initial real positions $z_1(0) = \alpha$ and $z_2(0) = \beta$ with corresponding velocities $\dot{z}_1(0) = 2i\,\alpha$ and $\dot{z}_2(0) = 2i\,\beta$, holds one of the some conditions*

1. *$\beta_1 = -\alpha$ inside of the geodesic circle, or its geodesic conjugated point $\beta_2 = \dfrac{R^2}{\alpha}$ outside of the geodesic circle.*
2. *For $\beta_3 = \dfrac{R(\alpha - R)}{\alpha + R}$ inside of the geodesic circle, or its geodesic conjugated point $\beta_4 = -\dfrac{R(R+\alpha)}{\alpha - R}$ outside of the geodesic circle,.*

Proof. In order of finding solutions for equation (35), we search those where both particles are sited in a same geodesic meridian along all the isometric action.

This is, without of generality we can put $z_1 = \alpha$ in the real axis, and $z_2 = \beta = \lambda\alpha$ for some suitable real value of λ, which will be founded with algebraic methods.

For this case we obtain that $R^2 + \bar{z}_1 z_2 = R^2 + \bar{z}_2 z_1$, and then, avoiding collisions and conjugated points, when we divide the left hand sides and the right hand ones of equation (35), it becomes into the equation

$$0 = \alpha[m_1(R^2 - \alpha^2)(R^2 + \lambda^2\alpha^2)^2 + m_2\lambda(R^2 - \lambda^2\alpha^2)(R^2 + \alpha^2)^2], \tag{36}$$

and for $\alpha \neq 0$, we obtain the simple equation in the unknown λ,

$$0 = m_1(R^2 - \alpha^2)(R^2 + \lambda^2\alpha^2)^2 + m_2\lambda(R^2 - \lambda^2\alpha^2)(R^2 + \alpha^2)^2. \tag{37}$$

For the algebraic equation (37) with equal masses, the solutions for the unknown λ are given by

$$\lambda_1 = -1, \quad \lambda_2 = \frac{R^2}{\alpha^2}, \quad \lambda_3 = \frac{R(\alpha - R)}{\alpha(\alpha + R)}, \quad \lambda_4 = -\frac{R(R+\alpha)}{\alpha(\alpha - R)},$$

which given the value of α, gives us the initial conditions for β:

$$\beta_1 = -\alpha, \quad \beta_2 = \frac{R^2}{\alpha}, \quad \beta_3 = \frac{R(\alpha - R)}{\alpha + R}, \quad \beta_4 = -\frac{R(R+\alpha)}{\alpha - R},$$

A simple analysis on the positions of such values for β proves the the corresponding positions in the items of the Lemma, which ends the proof. □

We obtain the main result of this subsection, the necessary and sufficient conditions for the existence of classes of Möbius elliptic solutions invariant under the Killing vector field (4) for the two body problem.

Theorem 2. *For the two body problem in $\mathbb{M}^2_R$ with fixed equal masses m we have the following class of Möbius elliptic solutions,*

1. *For the mass ratio $m < 2R^3$,*

- **a.** *There exists a unique initial condition $0 < \alpha_1 < (\sqrt{2}-1)R$ for one circular motion inside Ω_1, and for this condition there are two distinct non degenerate circular motions with real initial conditions,*
  - **i.** $\beta_{1,1} = -\alpha_1$ *on the same circle inside Ω_1.*
  - **ii.** $\beta_{1,2} = \dfrac{R^2}{\alpha_1}$ *for one circle sited in Ω_4.*
- **b.** *There exists a unique initial condition $(\sqrt{2}-1)R < \alpha_2 < R$ for a circular motion inside Ω_1 and for this condition, there are two distinct non degenerate circular motions with initial conditions,*
  - **i.** $\beta_{2,1} = -\alpha_2$ *on the same circle inside Ω_2.*
  - **ii.** $\beta_{2,2} = \dfrac{R^2}{\alpha_2}$ *sited in a different circle inside of Ω_3.*

2. *For the mass ratio $m < 2R^3$,*

- **a.** *There exists a unique initial condition $0 < \alpha_3 < (\sqrt{2}-1)R$ for one circular motion inside of Ω_1, and for this condition there exist two distinct non degenerate circular motions with real initial conditions,*
  - **i.** $\beta_{3,1} = \dfrac{R(\alpha_1 - R)}{\alpha_1 + R}$ *sited on different circle, but inside of Ω_1.*
  - **ii.** $\beta_{3,2} = \dfrac{R(\alpha_1 + R)}{R - \alpha_1}$ *sited on a different circle inside of Ω_4.*
- **b.** *There exists a unique initial condition $(\sqrt{2}-1)R < \alpha_4 < R$ for one circular motion inside of Ω_2, and for this condition there are two distinct non degenerate circular motions with real initial conditions,*
  - **i.** $\beta_{4,1} = \dfrac{R(\alpha_2 - R)}{\alpha_2 + R}$ *sited on different circle, but inside of Ω_2.*
  - **ii.** $\beta_{4,2} = \dfrac{R(\alpha_2 + R)}{R - \alpha_2}$ *sited on different circle and inside of Ω_3.*

3. *For the the mass ratio $m = 2R^3$, there exist the unique initial condition $\alpha_{tan} = (\sqrt{2}-1)R$ for a circular motion along T_S, and for this condition, there exist two distinct degenerate solutions with real initial conditions,*

- **a.** $\beta_{\tan,1} = -(\sqrt{2}-1)R$ *sited in the same circle T_S.*
- **b.** $\beta_{\tan,2} = \dfrac{R}{(\sqrt{2}-1)}$ *sited in the circle T_N.*

When the mass ratio $m > 2R^3$ there are not Möbius elliptic solutions for this problem.

Proof. We will use the necessary condition given by Lemma 2. In order of prove the sufficient condition, we separate the proof in all the cases.

1. a. For the cases $\beta = -\alpha$ and $\beta = \dfrac{R^2}{\alpha}$, when we substitute in any equation of system (35) we obtain the single and same equation,

$$\frac{|R^2-\alpha^2|\alpha}{(R^2+\alpha^2)^2} = \frac{\sqrt[3]{m}}{4\sqrt[3]{2}\,R^2} \tag{38}$$

which, since the solutions $\beta = -\alpha$ and $\beta = \dfrac{R^2}{\alpha}$ of this equation appear together with their geodesic conjugates, as can be directly checked, we will analyse equation (38) for $0 < \alpha < R$.

We define the smooth functions,

$$F(\alpha) = \frac{(R^2-\alpha^2)\alpha}{(R^2+\alpha^2)^2}, \quad G(\alpha) = \frac{\sqrt[3]{m}}{4\sqrt[3]{2}\,R^2}. \tag{39}$$

on the interval $[0, R]$.

The function F vanishes in the extremes of the interval and is increasing in the interval $(0, (\sqrt{2}-1)R)$, decreasing in $((\sqrt{2}-1)R, R)$, with an absolute maximum

$$F\left((\sqrt{2}-1)R\right) = \frac{1}{4R}. \tag{40}$$

This implies that, if

$$F\left((\sqrt{2}-1)R\right) - G\left((\sqrt{2}-1)R\right) = \frac{1}{4R} - \frac{\sqrt[3]{m}}{4\sqrt[3]{2}\,R^2} > 0 \tag{41}$$

or equivalently, $m < 2R^3$, the graphs of the functions F and G intersect transversally and in exactly two points, say $(\alpha_1, F(\alpha_1)) = (\alpha_1, G(\alpha_1))$ and $(\alpha_2, F(\alpha_2)) = (\alpha_2, G(\alpha_2))$ with the corresponding arguments $0 < \alpha_1 < (\sqrt{2}-1)R < \alpha_2 < R$, which define the initial positions for the aforementioned solutions **i** and **ii**. This proves the claim of items **1. a.** and **1. b**.

On the other hand, if $m = 2R^3$, the functions F and G intersect tangentially and in exactly the point $\left((\sqrt{2}-1)R, \dfrac{1}{4R}\right)$ which define the initial position for the respective degenerate solutions **a.** and **b.** of **3.** (see [17]).

2. a. If $\beta = -\dfrac{R(R-\alpha)}{R+\alpha}$ or $\beta = \dfrac{R(R+\alpha)}{R-\alpha}$, when we substitute in any equation of system (35) we obtain the equation

$$\frac{(R^2-\alpha^2)\,\alpha}{(R^2+\alpha^2)^2} = \frac{m}{8R^4}. \tag{42}$$

Since again the solutions $\beta = -\dfrac{R(R-\alpha)}{R+\alpha}$ and $\beta = \dfrac{R(R+\alpha)}{R-\alpha}$ of this equation appear together with their geodesic conjugates, as

can be again directly checked, we analyse equation (42) for the case $0 < \alpha < R$.

If we define the smooth functions $F(\alpha)$ and $G(\alpha) = \dfrac{m}{8R^4}$ as in the item **1.**, with the same analysis we obtain that

$$F\left((\sqrt{2}-1)R\right) - G\left((\sqrt{2}-1)R\right) = \frac{1}{4R} - \frac{m}{8R^4} > 0 \tag{43}$$

or equivalently, $m < 2R^3$, the graphs of the functions F and G intersect transversally and in exactly two points, say again $(\alpha_3, F(\alpha_3)) = (\alpha_3, G(\alpha_3))$ and $(\alpha_4, F(\alpha_4)) = (\alpha_4, G(\alpha_4))$ with the found arguments $0 < \alpha_3 < (\sqrt{2}-1)R < \alpha_4 < R$, which define the initial positions for the corresponding solutions **i** and **ii**. This proves the claims **2. a.** and **2. b.**

On the other hand, if $m = 2R^3$, the functions F and G intersect tangentially and in exactly the point $\left((\sqrt{2}-1)R, \dfrac{1}{4R}\right)$ which define the initial position for the corresponding degenerate solutions **a.** and **b.** of **3.** (see again [17]).

It is clear that for the mass ratio $m > 2R^3$ there are not Möbius elliptic solutions of the problem. This ends the proof. □

Corollary 4. *There are only teen different class of Möbius elliptic solutions for the 2-body problem in $\mathbb{M}^2_R$. Eight of them are non-degenerate while four of them are.*

The Theorem 2 proves the existence of new orbits on different hemispheres and shows a rich dynamics in this problem. In other words, by using the Möbius geometry we show new Möbius elliptic solutions for the n–body problem in $\mathbb{M}^2_R$. We have the following important result.

Corollary 5. *There are Möbius elliptic solutions for the n-body problem in $\mathbb{M}^2_R$ invariant under the Killing vector fields (3) and (7).*

4.1.2. **Möbius elliptic solutions for the Eulerian three-body problem in $\mathbb{M}^2_R$.** In [17] the authors found the so called elliptic Eulerian solutions for the initial positions $z_1(0) = \alpha, z_2(0) = 0, z_3(0) = \beta$ with α,β real numbers. They suppose that z_1 and z_3 have mass m, and z_2 has mass M.

The algebraic system of equations (27) becomes for this case,

$$\begin{aligned}
\frac{32R^6(\alpha^2-R^2)\alpha}{(R^2+\alpha^2)^4} &= \frac{-M\alpha}{|\alpha|^3} + \frac{m(\beta^2+R^2)^2(R^2+\beta\alpha)(\beta-\alpha)}{|\beta-\alpha|^3|R^2+\beta\alpha|^3},\\
0 &= \frac{\alpha(\alpha^2+R^2)^2}{|\alpha|^3} + \frac{\beta(\beta^2+R^2)^2}{|\beta|^3},\\
\frac{32R^6(\beta^2-R^2)\beta}{(R^2+\beta^2)^4} &= \frac{-M\beta}{|\beta|^3} + \frac{m(\alpha^2+R^2)^2(R^2+\alpha\beta)(\alpha-\beta)}{|\alpha-\beta|^3|R^2+\alpha\beta|^3}.
\end{aligned} \tag{44}$$

From the second equation in the system (44) is necessary that $\alpha\beta < 0$, and without lose of generality we can suppose that $\beta < 0 < \alpha$. Therefore such equation become into the one,

$$\beta\alpha^2 + (R^2 + \beta^2)\alpha + \beta R^2 = 0, \tag{45}$$

which has the solutions,

$$\begin{aligned} \alpha &= -\beta \quad \text{for} \quad |\beta| < R, \\ \alpha &= \frac{-R^2}{\beta} \quad \text{for} \quad |\beta| > R. \end{aligned} \tag{46}$$

Since the second equality is a geodesic conjugated algebraic solution for α, then unique solution is the antisymmetric $\alpha = -\beta$ inside the geodesic circle $|z| = R$.

We have the following result on this Möbius configuration of the Eulerian three body problem.

Corollary 6. (to Theorem 5.1 of [17]**)** *For the fixed mass with mass ratio $4R^3 > \frac{M}{2} + \frac{m}{4}$ there are only four distinct possible class of non degenerate solutions for the Eulerian Möbius three-body problem in $\mathbb{M}^2_R$ with one particle of mass M sited in the origin of coordinates and the other two of mass m sited in opposite sides of the same circle. There is one of them in each region Ω_i of $\mathbb{M}^2_R$. For the case $4R^3 = \frac{M}{2} + \frac{m}{4}$ are only two distinct degenerated solutions, one along T_S and the other along T_N. In the case $4R^3 < \frac{M}{2} + \frac{m}{4}$ such configuration does not exist.*

Proof. If we substitute $\alpha = -\beta$ both in the first as and the third equations of system (44), we obtain the same following equation

$$32\left(\frac{\alpha R^2(R^2 - \alpha^2)}{(R^2 + \alpha^2)^2}\right)^3 = M\left(\frac{R^2 - \alpha^2}{R^2 + \alpha^2}\right)^2 + \frac{m}{4}. \tag{47}$$

If α is a solution of the algebraic equation (47), a simple substitution proves that the geodesic conjugated real number $-\frac{R^2}{\alpha} < -R$ also is a solution of such that equation, and such number is outside of the geodesic circle $z = R$.

Therefore it will be sufficient with considering the problem for $0 < \alpha < R$, and for this we consider the smooth real functions

$$\begin{aligned} F(\alpha) &= 32\left(\frac{\alpha R^2(R^2 - \alpha^2)}{(R^2 + \alpha^2)^2}\right)^3, \\ G(\alpha) &= M\left(\frac{R^2 - \alpha^2}{R^2 + \alpha^2}\right)^2 + \frac{m}{4}, \end{aligned} \tag{48}$$

in the interval $[0, R]$.

For the first function F in the pair (48), we have that it is increasing in the interval $[0, (\sqrt{2}-1)R)$ and decreasing in the interval $((\sqrt{2}-1)R, R]$, having a maximum value $F((\sqrt{2}-1)R) = 4R^3$.

On the other hand, the second function G is strictly decreasing in the interval $[0, R]$ and has the value $G((\sqrt{2}-1)R) = \dfrac{M}{2} + \dfrac{m}{4}$.

Therefore, if we define the function $H(\alpha) = F(\alpha) - G(\alpha)$, then

$$
\begin{aligned}
H(0) &= F(0) - G(0) = -\left(\frac{M}{2} + \frac{m}{4}\right) < 0, \\
H((\sqrt{2}-1)R) &= F((\sqrt{2}-1)R) - G((\sqrt{2}-1)R) \\
&= 4R^3 - \left(\frac{M}{2} + \frac{m}{4}\right) > 0,
\end{aligned}
\tag{49}
$$

by hypothesis.

It follows that there exists $\alpha_1 \in (0, (\sqrt{2}-1)R)$ such $H(\alpha_1) = 0$, and this is a solution for (47).

Moreover, $F(R) = 0$ and $G(R) = \dfrac{m}{4}$, and then $H(R) = -\dfrac{m}{4} < 0$, which implies that there exists $\alpha_2 \in ((\sqrt{2}-1)R, R)$ such that $H(\alpha_2) = 0$, this is, α_2 is also solution of (47).

The transversal intersection of the smooth functions F and G on the interval $[0, R]$ for this case shows the uniqueness of such solutions, which determine the positions for the required solutions.

If the intersection is not transversal then we obtain the single point $\alpha = (\sqrt{2}-1)R$ where the functions F and G are tangential, which determine the position of the particle for obtaining the unique degenerated solution inside the geodesic circle.

The last claim is obvious and this ends the proof. □

4.1.3. **Möbius elliptic solutions in the four-body problem in $\mathbb{M}_R^2$.** We give an example for the the Four-body problem in $\mathbb{M}_R^2$, by considering four particles all of equal masses m, which are sited in the early on the ellipse $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$, with the initial conditions $z_1(0) = \alpha, z_2(0) = -\alpha, z_3(0) = i\beta, z_4(0) = -i\beta$, and with the condition $0 < \beta \leq \alpha$.

The algebraic system of equations (27), after reducing, are for this case,

$$
\begin{aligned}
\frac{32R^6(R^2-\alpha^2)\alpha}{(R^2+\alpha^2)^4} &= \frac{m(R^2+\alpha^2)^2(R^2-\alpha^2)}{4\alpha^2|R^2-\alpha^2|^3} + \frac{2\alpha m(R^2+\beta^2)^2(R^2-\beta^2)}{|\alpha+i\beta|^3|R^2+i\alpha\beta|^3}, \\
\frac{32R^6(R^2-\beta^2)\beta}{(R^2+\beta^2)^4} &= \frac{m(R^2+\beta^2)^2(R^2-\beta^2)}{4\beta^2|R^2-\beta^2|^3} + \frac{2\beta m(R^2+\alpha^2)^2(R^2-\alpha^2)}{|\alpha+i\beta|^3|R^2+i\alpha\beta|^3}.
\end{aligned}
\tag{50}
$$

It is clear that $\alpha = \beta$ satisfies the pair of equations (50), which implies the following result (see [1] and [6]).

Corollary 7. (to Theorem 1 of [6]**)** *For the four-body problem with equal masses on the same circle and with the configuration of square, up a geodesic conjugation, there are only six class of solutions. Four are non degenerate with two inside the geodesic circle and two outside. The other two are degenerate, one is sited inside the geodesic circle and the other is outside.*

Proof. The proof follows the same methodology as in Theorem 2 and Corollary 6.

If we put $\beta = \alpha$ in any of the equations (50), we obtain, after reducing it,

$$\frac{32R^6\alpha^3}{m(R^2+\alpha^2)^6} = \frac{1}{4|R^2-\alpha^2|^3} + \frac{1}{\sqrt{2\,(\alpha^4+R^4)^3}} \tag{51}$$

If α is a solution of the algebraic equation (51), another simple substitution proves that the geodesic conjugated real number $-\dfrac{R^2}{\alpha} < -R$ is also a solution of such that equation, and such number is outside of the geodesic circle $z = R$.

Therefore it will be sufficient with considering the problem for $0 < \alpha < R$, and for this we consider the smooth real functions

$$\begin{aligned} F(\alpha) &= \frac{32R^6\alpha^3}{m(R^2+\alpha^2)^6}, \\ G(\alpha) &= \frac{1}{4(R^2-\alpha^2)^3} + \frac{1}{\sqrt{2\,(\alpha^4+R^4)^3}}, \end{aligned} \tag{52}$$

in the interval $[0, R)$.

For the first function F in the pair (52), we have that it is increasing in the interval $\left[0, \dfrac{\sqrt{3}R}{3}\right)$ and decreasing in the interval $\left(\dfrac{\sqrt{3}R}{3}, R\right]$, having a maximum value $F\left(\dfrac{\sqrt{3}R}{3}\right) = \dfrac{81\sqrt{3}}{128mR^3}$.

On the other hand, the second function G is also increasing in such interval with a singular point in $\alpha = R$ and has the value $G\left(\dfrac{3R}{\sqrt{3}}\right) = \dfrac{27}{4R^6}\left(\dfrac{1}{8} + \dfrac{1}{5\sqrt{5}}\right)$.

Therefore, if we define again the function $H(\alpha) = F(\alpha) - G(\alpha)$, then

$$
\begin{aligned}
H(0) &= F(0) - G(0) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)\frac{1}{R^6} < 0,\\
H\left(\frac{\sqrt{3}R}{3}\right) &= F\left(\frac{\sqrt{3}R}{3}\right) - G\left(\frac{\sqrt{3}R}{3}\right)\\
&= \frac{81\sqrt{3}}{128mR^3} - \frac{27}{4R^6}\left(\frac{1}{8} + \frac{1}{5\sqrt{5}}\right) > 0,
\end{aligned}
\tag{53}
$$

and for the necessary mass condition $R^3 > \dfrac{128m}{81\sqrt{3}}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{4}\right)$ there exists $\alpha_1 \in \left(0, \dfrac{\sqrt{3}R}{3}\right)$ such $H(\alpha_1) = 0$, and this is, as before, a solution for (51).

On the other hand, since $F(\alpha)$ is strictly increasing in the interval $\left(\dfrac{\sqrt{3}R}{3}, R\right)$, $G(\alpha)$ is strictly decreasing in such interval and

$$
\begin{aligned}
\lim_{\alpha\to R^-} F(\alpha) &= \frac{1}{2mR^3},\\
\lim_{\alpha\to R^-} G(\alpha) &= +\infty,
\end{aligned}
\tag{54}
$$

then there exists $\theta \in \left(\dfrac{\sqrt{3}R}{3}, R\right)$ such that $H(\theta) < 0$. Therefore there exists $\alpha_2 \in \left(\dfrac{\sqrt{3}R}{3}, \theta\right)$ such $H(\alpha_2) = 0$. This is, α_2 is a second solution of (51).

Once again, as before, the transversality of functions F and G on the interval $[0, R)$ shows the uniqueness of such solutions in this interval, which proves that for the necessary mass condition $R^3 > \dfrac{128m}{81\sqrt{3}}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{4}\right)$ there are two solutions for the equation (51) in the interval $(0, R)$.

On the other hand, for the mass condition $\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{4}\right)m > \dfrac{81\sqrt{3}R^3}{128}$, there are not intersection between the functions F and G.

Therefore, from continuity of the problem respect to parameters, there is at least one mass condition such that the intersection between F and G is in a single point.

In order of finding this sufficient condition for the having elliptic Möbius solutions for this problem, it is necessary to find the tangential argument

$\alpha \in \left(0, \dfrac{\sqrt{3}R}{3}\right)$ such that

$$\begin{aligned} F(\alpha) - G(\alpha) &= 0, \\ F'(\alpha) - G'(\alpha) &= 0, \end{aligned} \tag{55}$$

in terms of the parameters R, m, say $\alpha_{\tan} = \alpha(R, m)$. The existence of this tangential point $\alpha_{\tan}$ follows from the concavities of the functions $F(\alpha)$ and $G(\alpha)$. The uniqueness follows from the fact that the function $F(\alpha)$ has only one inflection point in the interval $\left(0, \frac{\sqrt{3}R}{3}\right)$, and the existence of more tangential points must imply a bigger number of inflection points, as a straightforward calculus shows.

Therefore, we obtain the following cases.

(1) For the mass ratio $F(\alpha_{\tan}) - G(\alpha_{\tan}) > 0$ there exist two solutions α_1, α_2 of equation (51) in the interval $(0, R)$ such that $0 < \alpha_1 < \alpha_{\tan} < \alpha_2$. Such solutions are the initial conditions for two non degenerate solutions of this four-body problem.

(2) For the mass ratio $F(\alpha_{\tan}) - G(\alpha_{\tan}) = 0$ there exist a unique solution $\alpha = \alpha_{\tan}$ of equation (51) in the interval $(0, R)$. Such solution is the initial conditions for one degenerate solution of this four-body problem.

(3) For the mass ratio $F(\alpha_{\tan}) - G(\alpha_{\tan}) < 0$ there exist are not solution for this four-body problem.

This ends the proof. □

5. Möbius hyperbolic solutions

We show the Möbius solutions generated by the one dimensional hyperbolic subgroup of $\mathbf{Mob}_2\left(\mathbb{M}_R^2\right)$ in the Iwasawa decomposition (12).

In one Riemannian manifold, a pair of points p_1 and p_2 are geodesic conjugated if there exists at least a pair of different geodesics joining them. It is well known that in $\mathbb{M}_R^2$ any pair of antipodal points $z_k = \dfrac{-R^2}{|z_j|^2} z_j$ are geodesic conjugated and the whole space is foliated by all geodesic curves passing through such pair of points. The set of all such curves is called the *geodesic conjugated class foliation*.

Definition 5. *A solution $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ of equation (21) is called homothetic if all the particles move on curves whose path belong to the same geodesic conjugated class foliation.*

Proceding as in Lemma 1, up an isometry, we can assume that one point is the origin of coordinates $z = 0$ with conjugated point $z = \infty$, and the geodesic foliation of $\mathbb{M}_R^n$ is the set of straight lines passing through such point called *meridians*.

We remark that since the geodesics are always parametrized such that their tangent vectors have constant speed (see [7]), then one particle moving along one homothetic solution not necessarily does it in a geodesic way.

Among the whole set of homothetic solutions of (21) is the subclass generated by the action of the one-parametric subgroup of hyperbolic Möbius transformations (8).

Definition 6. *An homothetic solution* $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ *of equation (21) is called Möbius hyperbolic if it is invariant under the vector field (9).*

We state the following result for this type of dolutions in the n-body problem, which follows from direct substitutions.

Theorem 3. *Let be* n *point particles with masses* $m_1, m_2, \cdots, m_n > 0$ *moving in* $\mathbb{M}^2_R$. *An equivalent condition for* $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ *to be a Möbius hyperbolic solution of (21), is that the coordinates satisfy the system of rational functional functions (depending on the time* t*),*

$$\frac{8\,R^6(R^2 - |z_k|^2)z_k}{(R^2 + |z_k|^2)^4} = \sum_{j=1, j\neq k}^{n} \frac{m_j\,(|z_j|^2 + R^2)^2(R^2 + \bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3\,|R^2 + \bar{z}_j z_k|^3} \tag{56}$$

and the velocity in each particle is given by the relation $\dot{z}_k(t) = z_k(t)$*, for* $k = 1, 2, \cdots, n$. *All the solutions* $z_k = z_k(t)$ *are backward asymptotic to the origin of coordinates, which implies that there are not collisions between the particles.*

Proof. If we derive the vector field (9), we obtain $\ddot{z}_k = z_k$, which when is substituted into equations of motion (21) allows to the system (56). This ends the proof of the Theorem. □

For the two body problem, if we suppose that two particles with masses m_1 and m_2 and positions $z_1(t)$ and $z_2(t)$ in $\mathbb{M}^2_R$ are moving as a Möbius hyperbolic solution, in [18] is shown that, the masses of the particles are equal, if and only if, $z_1(t) = -z_2(t)$. It follows that, up an isometry, for the two-body problem with equal masses there is only a single class of Möbius hyperbolic solutions.

In [4], the authors show that a necessary and sufficient condition for having a *Lagrangian* homothetic solution in the 3–body problem in the unitary sphere embedded in $\mathbb{R}^3$, is that the configuration be always an equilateral triangle and that the masses (always sited in the same horizontal plane intersecting the sphere) be equal. Therefore, in order to study in beyond this kind of motion in $\mathbb{M}^2_R$, for the Möbius hyperbolic solutions case, it is enough to analyse the case of equal masses.

6. Möbius nilpotent parabolic solutions

Finally, we show the Möbius nilpotent parabolic solutions corresponding to the second factor N in the Iwasawa decomposition (12), and associated to the one dimensional subgroup (10) of $\mathbf{Mob}_2\,(\mathbb{M}^2_R)$.

Definition 7. *A solution* $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ *of equation (21) is called Möbius nilpotent parabolic if it is invariant under the parabolic vector field (11).*

We can now state the following result whose proof follows again by straightforward substitutions.

Theorem 4. *Let be* n *point particles with masses* $m_1, m_2, \cdots, m_n > 0$ *moving in* $\mathbb{M}^2_R$. *An equivalent condition for* $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ *to be a to be a Möbius nilpotent parabolic solution of system (21) is that the coordinate functions, depending of the time* t, *satisfy the functional (also depending on the time* t*) equations*

$$-\frac{16R^6\bar{z}_k}{(R^2+|z_k|^2)^4} = \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{m_j\,(R^2+|z_j|^2)^2(R^2+z_k\bar{z}_j)(z_j-z_k)}{|z_j-z_k|^3\,|R^2+\bar{z}_j z_k|^3}, \tag{57}$$

and with corresponding velocities $\dot{z}_k(t) = 1$, *for* $k = 1, \ldots, n$.

In [5] the authors have shown that there are not Möbius nilpotent parabolic solutions for the n–body problem with negative curvature. In [19] it is shown the non existence of true Möbius parabolic solutions for the n-body problem in $\mathbb{H}^2_R$. We remark that in $\mathbb{M}^2_R$ these two types of conic motions coincide (although they not be isometries as in $\mathbb{H}^2_R$), and unfortunately the same happens for their existence in this positive case.

Corollary 8. *There are no one class of Möbius nilpotent parabolic solutions for the* n*–body problem in* $\mathbb{M}^2_R$.

Proof. Let be n point particles of masses $m_1, \ldots, m_n$ moving on $\mathbb{M}^2_R$ with total vector position $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))$, with $z_k = z_k(t)$ satisfying equations (57).

Since due to the action of the vector field (11) the real part of the solutions become positive for suitable large values of time, then applying a translation to the whole set of solutions if it is necessary, we can assume that the k-th particle reaches the imaginary axis. This is, we assume $\mathrm{Re}\,(z_k) = 0$, and $\mathrm{Re}\,(z_j) \geq 0$ for all $z_j \neq z_k$ with $j = 1, \ldots, n$.

Therefore, the real parts in each side of system (57) are

$$\begin{aligned} 0 &= -\mathrm{Re}\left(\frac{16R^6\bar{z}_k}{(R^2+|z_k|^2)^4}\right) \\ &= \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{m_j\,(R^2+|z_j|^2)^2}{|z_j-z_k|^3\,|R^2+\bar{z}_j z_k|^3}\,\mathrm{Re}\,[(R^2+z_k\bar{z}_j)(z_j-z_k)] \\ &= \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{m_j\,(R^2+|z_j|^2)^2}{|z_j-z_k|^3\,|R^2+\bar{z}_j z_k|^3}\,(R^2+|z_k|^2)\mathrm{Re}\,[z_j]. \end{aligned} \tag{58}$$

It follows, from the chain of equalities (58) that the whole set of particles must be also located on the imaginary axis, that is, $\mathrm{Re}\,(z_j(0)) = 0$ for all $j = 1, 2, \cdots, n$.

Therefore, if we put $z_l(t) = t + i\beta_l$ for the position of the l-th particle, then we obtain the equalities

$$(R^2 + z_k \bar{z}_j)(z_j - z_k) = t\,(\beta_j - \beta_k)^2 + \left(t^2 + \beta_j\beta_k + R^2\right)(\beta_j - \beta_k)\;i. \tag{59}$$

When we substitute equations (59) in the system (57), we obtain for the real parts the relations

$$-\frac{16R^6}{(R^2 + |z_k|^2)^4} = \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{m_j\,(R^2 + |z_j|^2)^2}{|z_j - z_k|^3\,|R^2 + \bar{z}_j z_k|^3}\;(\beta_j - \beta_k)^2\,, \tag{60}$$

for $k, j = 1, \cdots, n$, and for all $t \in \mathbb{R}$.

System (60) never holds, since if we avoid collisions and geodesic conjugated points, the left hand side is always negative, whereas the right hand side is positive. This contradiction proves the Theorem. $\square$.

7. Conclusions

As in [19], we have that unique Möbius solutions for the n-body problem in $\mathbb{M}^2_R$ are either the elliptic, the hyperbolic or the composition of them (called loxodromic). Again, as we have pointed there, for the complete study of this type of solutions, it is sufficient with considering the Cartan-Haussdorf decomposition KBK of $SL(2,\mathbb{C})$. Also as in that reference, in words of the classical mechanics we have the following result.

Corollary 9. *The unique relative equilibria for the two dimensional positively curved n-body problem are those Möbius elliptic solutions.*

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